Note on anomalies in field theories

Chang-Yong Liu
College of Science, Northwest A&F University, Yangling, Shaanxi 712100, China
E-mail: liuchangyong@nwsuaf.edu.cn

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Abstract

In this note, we investigate the anomalies in field theories. The results of the anomalies through Feynman diagrams calculation are multi-valued function. These single-valued branches of multi-valued function are related to the bound states of neutral pseudoscalar mesons. Adding these bound state contributions, we obtain a new anomaly free condition that all the external particles are on-shell and find the non-perturbative mass spectrum of neutral pseudoscalar mesons those are $m_P^2(n) = 8n\pi^2 m^2$. Generalize this formula, we obtain the mass of η' meson ($m_{\eta'} = 961\,\mathrm{MeV}$) which is almost the same as the experimental value ($m_{\eta'}^{\mathrm{ex}} \approx 958\,\mathrm{MeV}$). We also discuss the anomaly in 1+1 dimensional QFT.

Keywords: Anomaly, Quantum Field Theory

Contents

1 Introduction

Anomaly plays an important role in Quantum Field Theory (QFT) [1]. An anomaly arises when a symmetry group G which leaves the classical action invariant violated in the full quantum theory. The primary example is the anomaly for the decay rate $\pi^0 \to \gamma\gamma$. The decay rate confused people for many years. A gauge-invariant result was propsed by Steinberger [2] and Schwinger [3]. Steinberger considered a field theory in which pions are coupled to massive fermions which are proton and neutron at the time, although they could just as well have been quarks. The connection to the anomalous symmetries was finally understood in 1969 due to the work of Adler [4], Bell and Jackiw [5]. The anomaly was calculated through Feynman diagrams. Fujikawa [6] later pointed out that anomalies arise when a measure in the path integral is not invariant under the group G which leaves the classical action invariant. The anomaly have many other applications. They presents a successful explanation for $U(1)$ problem [8]. As we know, the lightest three neutral pseudoscalar mesons are π^0, η and η' (Table 1). The Gell-Mann and Levy [9] built a Lagrangian with $SU(2) \times SU(2)$ chiral symmetry and a nucleon mass if the chiral symmetry was spontaneously broken. In this model, the pion is interpreted as a Goldstone boson. This model can be extended to encompass an $SU(3) \times SU(3)$ chiral symmetry. We encounter the $U(1)$ problem that the η' is very heavy. The Witten-Veneziano relation [10, 11] gives a way to solve the $U(1)$ problem. The $U(1)$ problem can not be solved within perturbative framework. A non-perturbative method is required.

Pseudoscalar meson	wave function	mass (experimental value)
π^0	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$	135 MeV
η	$\frac{1}{\sqrt{6}}(u\bar{u} - d\bar{d} - 2s\bar{s})$	548 MeV
η'	$\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$	958 MeV

Table 1: Pseudoscalar meson masses (e. g. text book [7]).

In this note, we study the anomalies in field theories. The new result is that there are many single-valued branches of multi-valued function which are not considered before. These single-valued branches have physical consequence. So we reconsider the well-known results of anomalies. Our results also give a

new method to study the mass of η'. In our previous work [12,13], we have connected the single-valued branch of multi-valued function in correlation function with bound states. In order to study the bound states, we define the integral of a complex function $f(z)$ along a smooth contour $C[a,b]$ in a complex plane. Suppose the function $f(z)$ have pole and branch cut (Figure 1), then the integral of $f(z)$ along the contour $C[a,b]$ can be expressed as

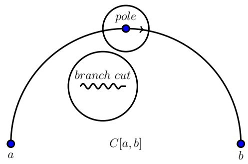

Figure 1: A smooth contour $C[a,b]$ in complex plane staring from a to b.

$$\int_{C[a,b]} f(z)dz = P\int_a^b f(z)dz + \sum n_i \oint_{C_i} f(z)dz. \tag{1}$$

Where the C_i is a closed curve circling the pole or branch cut (Figure 1). The $P\int_a^b f(z)dz$ denotes the principal value which takes value in main single-valued branch. The winding number $n_i \in Z$ is the contour circling n_i times around the pole or branch cut.

The paper is organized as follows. In Section 2, we study the axial (ABJ) anomaly in (3+1) dimensions. We obtain a new anomaly free condition and find the non-perturbative mass spectrum of neutral pseudoscalar mesons. In Section 3, we discuss the anomaly in 1+1 dimensional QFT. We end with the conclusions.

2 Axial (ABJ) anomaly in (3+1) dimensions

We first review the well known of axial (ABJ) anomaly [4,5] in (3+1) dimensions. Then we present our new formula. The 3+1 dimensional QED Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi. \tag{2}$$

The vector and the axial vector currents are

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \quad j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi. \tag{3}$$

From the classical equations of motion, the divergences of two currents are

$$\partial_\mu j^\mu = 0, \quad \partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi. \tag{4}$$

Thus, classically the vector symmetry is exactly conserved, while the chiral symmetry is only conserved in the massless limit. We calculate the one-loop amplitude to describe how the classical equations (4) are

modified within quantum theory. We consider the correlation function $\langle j^{\rho 5} j^{\mu} j^{\nu} \rangle$. In momentum space, at 1-loop the correlation function is (Fig. 2)

$$iM_5^{\rho\mu\nu} = -\int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}[\frac{i}{\not p - m}\gamma^{\mu}\frac{i}{\not p + \not k_1 - m}\gamma^{\rho}\gamma^5\frac{i}{\not p - \not k_2 - m}\gamma^{\nu} + \frac{i}{\not p - m}\gamma^{\nu}\frac{i}{\not p + \not k_2 - m}\gamma^{\rho}\gamma^5\frac{i}{\not p - \not k_1 - m}\gamma^{\mu}].$$

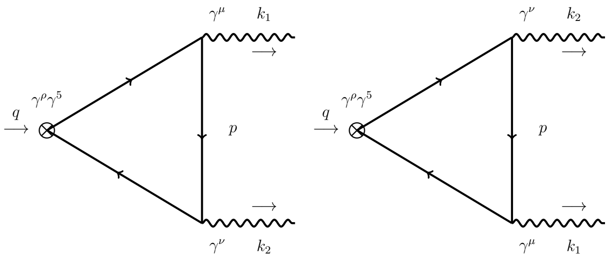

Figure 2: One-loop diagrams for $iM_5^{\rho\mu\nu}$.

Contracting the axial current with its momentum q^ρ yields [14, 15]

$$q^{\rho}M_5^{\rho\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}[\frac{\not p + m}{p^2 - m^2}\gamma^{\mu}\frac{\not p + \not k_1 + m}{(p+k_1)^2 - m^2}\not q\gamma^5\frac{\not p - \not k_2 + m}{(p-k_2)^2 - m}\gamma^{\nu} + \left(\begin{array}{c}\mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2\end{array}\right)].$$

The integrand is linear divergence. This is a very subtle point which need carefully treatment. The result is

$$\begin{aligned} q^{\rho}M_5^{\rho\mu\nu} &= \frac{m^2}{2\pi^2}\epsilon^{\mu\nu}{}_{\rho\sigma}k_1^{\rho}k_2^{\sigma}\int_0^1 dx\int_0^{1-x}dy\frac{1}{m^2 - xk_1^2 - yk_2^2 + (xk_1 - yk_2)^2} - \frac{1}{4\pi^2}\epsilon^{\mu\nu}{}_{\rho\sigma}k_1^{\rho}k_2^{\sigma} \\ &+ \left(\begin{array}{c}\mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2\end{array}\right). \end{aligned} \tag{5}$$

In massless limit, we obtain

$$q^{\rho}M_5^{\rho\mu\nu} = -\frac{1}{2\pi^2}\epsilon^{\mu\nu}{}_{\rho\sigma}k_1^{\rho}k_2^{\sigma} \neq 0.$$

So that the axial current is not conserved in massless limit. This is the ABJ anomaly. Our new result is that there are many single-valued branches of multi-valued function which are not considered before. These single-valued branches of multi-valued function in correlation function are connect with bound states. In physical process, the external fields are on-shell, those are $k_1^2 = k_2^2 = 0$, $(k_1+k_2)^2 = 2k_1 \cdot k_2 = q^2$. Then the expression (5) becomes

$$q^{\rho}M_5^{\rho\mu\nu} = \frac{m^2}{2\pi^2}\epsilon^{\mu\nu}{}_{\rho\sigma}k_1^{\rho}k_2^{\sigma}\int_0^1 dx\int_0^{1-x}dy\frac{1}{m^2 - xyq^2} - \frac{1}{4\pi^2}\epsilon^{\mu\nu}{}_{\rho\sigma}k_1^{\rho}k_2^{\sigma} + \left(\begin{array}{c}\mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2\end{array}\right).$$

According to the formula (1), the integral $\int_0^1 dx \int_0^{1-x} dy \frac{1}{m^2-xyq^2}$ can be calculated as

$$\begin{aligned}\int_0^1 dx \int_0^{1-x} dy \frac{1}{m^2-xyq^2} &= \frac{1}{q^2}\int_0^1 \frac{1}{x}\ln\frac{1}{1-x(1-x)\frac{q^2}{m^2}}dx \\ &= P\frac{1}{q^2}\int_0^1 \frac{1}{x}\ln\frac{1}{1-x(1-x)\frac{q^2}{m^2}}dx + \frac{1}{q^2}[2\pi ik\ln\frac{1+\sqrt{1-\frac{4m^2}{q^2}}}{1-\sqrt{1-\frac{4m^2}{q^2}}} + (2\pi i)^2 kl]\end{aligned}$$

Where the l and k are $l \in Z$ and $k \in Z$. Then we get

$$\begin{aligned} q^\rho M_5^{\rho\mu\nu} &= \frac{m^2}{2\pi^2}\epsilon^{\mu\nu}{}_{\rho\sigma}k_1^\rho k_2^\sigma[\frac{1}{q^2}P\int_0^1 \frac{1}{x}\ln\frac{1}{1-x(1-x)\frac{q^2}{m^2}}dx + \frac{1}{q^2}2\pi ik\ln\frac{1+\sqrt{1-\frac{4m^2}{q^2}}}{1-\sqrt{1-\frac{4m^2}{q^2}}} + \frac{1}{q^2}(2\pi i)^2 n] \\ &- \frac{1}{4\pi^2}\epsilon^{\mu\nu}{}_{\rho\sigma}k_1^\rho k_2^\sigma + \begin{pmatrix}\mu\leftrightarrow\nu \\ k_1\leftrightarrow k_2\end{pmatrix}. \end{aligned} \tag{6}$$

Where we have defined the $n = kl \in Z$. Generally, the anomalies must vanish for a local gauge symmetry. Traditionally, the anomalous contributions coming from various multiplets cancel completely making the gauge current anomaly free. That is the $\mathcal{A}^{abc} = \mathrm{tr}[t^a\{t^b, t^c\}] = 0$ is a fundamental consistency condition for chiral gauge theories [16]. This is very important in building models of particle interactions. In our approach, we use the extra term $\frac{m^2}{2\pi^2}\epsilon^{\mu\nu}{}_{\rho\sigma}k_1^\rho k_2^\sigma[-\frac{4\pi^2}{q^2}n]$ in equation (6) to cancel the anomaly term $-\frac{1}{4\pi^2}\epsilon^{\mu\nu}{}_{\rho\sigma}k_1^\rho k_2^\sigma$. This lead to the anomaly free condition

$$\boldsymbol{q^2 = 8n\pi^2 m^2}, \quad n \in N. \tag{7}$$

We argue that this is the on-shell condition for external neutral pseudoscalar meson $P \in \{\pi^0, \eta, \eta'\}$. That is the q^2 is related to the neutral pseudoscalar meson mass m_P for a quantum number n, which is $q^2(n) = m_P^2(n)$. Their masses fall on a straight line, which is similar to the Regge trajectory [17]. Then the anomaly free condition is that all external fields are on-shell. The decay $P \to 2\gamma$ corresponds to the transition $\mid n\rangle \to \mid 0\rangle$, where the $\mid n\rangle$ denotes the n-th bound states of neutral pseudoscalar meson. On the other hand, the anomaly free condition give the mass spectrum of neutral pseudoscalar mesons. To illustrate this, we discuss the mass of neutral pseudoscalar meson η'. The wave function of meson P can be expressed as $|P\rangle = a_P|u\bar{u}\rangle + b_P|d\bar{d}\rangle + c_P|s\bar{s}\rangle$, where $a_P^2 + b_P^2 + c_P^2 = 1$ (Tabel 1). We denote the masses of u, d and s quarks as m_u, m_d and m_s in pseudoscalar meson. To discuss the neutral pseudoscalar meson in real world, we generalize the results (7) to three quarks. Suppose the meson wave function to be $|P\rangle = a_P|u\bar{u}\rangle + b_P|d\bar{d}\rangle + c_P|s\bar{s}\rangle$, then the mass of the ground state ($n = 1$) is

$$m_P = 2\sqrt{2}\pi(a_P^2 m_u + b_P^2 m_d + c_P^2 m_s), \qquad P \in \{\pi^0, \eta, \eta'\}. \tag{8}$$

We deduce the mass $m_{\eta'}$ from the masses of m_{π^0} and m_η in the following way by using the equation (8)

$$\boxed{m_{\pi^0}, m_\eta} \xrightarrow{(8)} \boxed{m_u, m_d, m_s} \xrightarrow{(8)} \boxed{m_{\eta'}}. \tag{9}$$

We first compute the masses m_u, m_d and m_s from the experimental data of π^0 and η. Then we can obtain the mass of η'. From equation (8), we find that we need to exchange the wave function of η with the wave function of η' in Tabel 1. We put the new identification in Tabel 2.

Pseudoscalar meson	wave function
π^0	$\frac{1}{\sqrt{2}}(u\bar{u}-d\bar{d})$
η	$\frac{1}{\sqrt{3}}(u\bar{u}+d\bar{d}+s\bar{s})$
η'	$\frac{1}{\sqrt{6}}(u\bar{u}-d\bar{d}-2s\bar{s})$

Table 2: We exchange the wave function of η with the wave function of η' in Table 1 .

From the equation (8), we find

$$\begin{cases} \frac{1}{2}(m_u+m_d)2\sqrt{2}\pi = 135\,\text{MeV} \quad (\pi^0), \\ \frac{1}{3}(m_u+m_d+m_s)2\sqrt{2}\pi = 548\,\text{MeV} \quad (\eta). \end{cases}$$

This gives the masses of m_u+m_d and m_s

$$\begin{cases} m_u+m_d = 30.4\,\text{MeV}, \\ m_s = 154.6\,\text{MeV}. \end{cases}$$

Because quarks are confined in QCD, their masses cannot be directly measured. There are different mass definitions of Quarks. The popular scheme is the $\overline{\text{MS}}$ scheme. The running masses $m_a(\mu)(a=u,d,s,\cdots)$ decrease with increasing μ. These properties illustrate that the above result of quark masses is correct in neutral pseudoscalar mesons. Applying the equation (8), we obtain the theoretical value of η' which is $m_{\eta'}=961\,\text{MeV}$. This is almost the same as the experimental value $m^{\text{ex}}_{\eta'}=958\,\text{MeV}$ (Table 1).

We now to discuss the massless limit. When the m^2 approach 0, we require the m^2 and q^2 satisfy the anomaly free condition (7) that is

$$\lim_{(m^2,q^2)\to(0,0)} \frac{q^2}{m^2} = 8n\pi^2. \tag{10}$$

Then the axial anomaly is free in the massless limit. We conclude that conservation of the axial current is restored if all the external particles are on-shell by adding the bound state contributions. In next section, we will consider the anomaly in 1+1 dimensional QFT.

3 Anomaly in 1+1 dimensional QFT

The 1+1 dimensional QED Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi. \tag{11}$$

We can choose the two Dirac matrices as

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{12}$$

The γ^5 is the product of the Dirac matrices

$$\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

The vector j^μ and the axial vector $j^{\mu 5}$ currents are the same as those in four dimensional QED (3,4). We calculate the one-loop amplitude involving a photon and an axial vector current (Fig. 3) to describe how the classical equations (4) are modified with quantum theory.

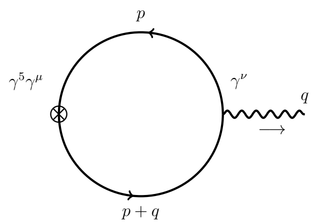

Figure 3: The amplitude with an axial vector current and a photon.

The Fourier transform of the matrix element $\langle 0|T(j_5^\mu(x)A^\nu(0))|0\rangle$ is given by

$$I^{5\mu\nu} = -e^2 \int \frac{d^2p}{(2\pi)^2} \mathrm{Tr} \frac{\gamma^5\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}+\not{q}+m)}{(p^2-m^2)[(p+q)^2-m^2]}. \tag{14}$$

The two dimensional Dirac matrices obey the identity

$$\gamma^5\gamma^\mu = \epsilon^{\mu\nu}\gamma_\nu. \tag{15}$$

Then the $I^{5\mu\nu}$ can be expressed as

$$\begin{aligned} I^{5\mu\nu} &= -e^2\epsilon^{\mu\rho} \int \frac{d^2p}{(2\pi)^2} \mathrm{Tr} \frac{\gamma_\rho(\not{p}+m)\gamma^\nu(\not{p}+\not{q}+m)}{(p^2-m^2)[(p+q)^2-m^2]} \\ &= \epsilon^{\mu\rho} i\Pi_{2\rho}{}^{\nu}. \end{aligned} \tag{16}$$

Where the $i\Pi_2^{\mu\nu}$ is the lowest-order vacuum polarization of QED [18], which is

$$\begin{aligned} i\Pi_2^{\mu\nu} &= -4ie^2(q^2g^{\mu\nu} - q^\mu q^\nu) \int_0^1 dx \int \frac{d^2p_E}{(2\pi)^2} \frac{x(1-x)}{(p_E^2+m^2-x(1-x)q^2)^2} \\ &= -i(q^2g^{\mu\nu} - q^\mu q^\nu)\frac{e^2}{\pi} \int_0^1 dx \frac{x(1-x)}{m^2-x(1-x)q^2}. \end{aligned} \tag{17}$$

Finally, we obtain

$$q_\mu I^{5\mu\nu} = -iq^2 q_\mu \epsilon^{\mu\nu} \frac{e^2}{\pi} \int_0^1 dx \frac{x(1-x)}{m^2-x(1-x)q^2}. \tag{18}$$

We now consider the massless limit which is known as the Schwinger model [19]. The equation (18) in massless limit gives the anomaly term

$$q_\mu I^{5\mu\nu} = -iq_\mu\epsilon^{\mu\nu}\frac{e^2}{\pi}. \tag{19}$$

In physical process, the external fields are on-shell ($q^2 = 0$). There is a subtle point. Taking the $q_\mu I^{5\mu\nu}$ as the function of q^2 and m^2, we find

$$\lim_{m^2\to 0}\lim_{q^2\to 0} q_\mu I^{5\mu\nu} \neq \lim_{q^2\to 0}\lim_{m^2\to 0} q_\mu I^{5\mu\nu}. \tag{20}$$

Then the function $q_\mu I^{5\mu\nu}(q^2, m^2)$ is actually undefined at the point $(q^2, m^2) = (0, 0)$. We now consider the pole contributions. According to the formula (1), we get

$$q_\mu I^{5\mu\nu} = -iq_\mu \epsilon^{\mu\nu} \frac{e^2}{\pi}(-1 + \frac{4m^2}{q^2} \frac{\arctan[\frac{1}{\sqrt{-1+\frac{4m^2}{q^2}}}]}{\sqrt{-1+\frac{4m^2}{q^2}}} + \frac{4m^2}{q^2} \frac{n\pi}{\sqrt{-1+\frac{4m^2}{q^2}}}). \tag{21}$$

As the same argument as axial anomaly in (3+1) dimensions, we find the anomaly free condition

$$-1 + \frac{4m^2}{q^2} \frac{n\pi}{\sqrt{-1+\frac{4m^2}{q^2}}} = 0. \tag{22}$$

The solution of the equation (22) is

$$q^2 = 2(1 \pm \sqrt{1 - 4n^2\pi^2})m^2, \quad n \in Z. \tag{23}$$

Where the q^2 take complex values for $m \neq 0$. Then the 1+1 dimensional pseudoscalar meson is different with 3+1 dimensional one. The m^2 and q^2 need to satisfy the relation (23) in the limit $(q^2, m^2) \to (0, 0)$, then the equation (21) becomes

$$\lim_{(m^2,q^2)\to(0,0)} q_\mu I^{5\mu\nu} = 0. \tag{24}$$

We obtain the axial anomaly is free if the all external particles are on-shell by taking account of the relation (23) in massless limit.

4 Conclusions and Discussions

In this note, we have studied the anomalies in field theories. The pole contributions in the calculation of Feynman diagrams lead to the multi-valued function. Similar to our previous work [12, 13], these single-valued branches of multi-valued function are related to the bound states of pseudoscalar mesons. From this, we cancelled the anomaly term with the bound state contribution. We present a new anomaly free condition that all the external particles are on-shell. We found that the non-perturbative mass spectrum of neutral pseudoscalar mesons are $m_P^2(n) = 8n\pi^2 m^2$. This formula can be used to study the mass of η' meson. We obtained the mass of η' meson ($m_{\eta'} = 961\,\text{MeV}$) which is almost the same as the experimental value ($m_{\eta'}^{\text{ex}} \approx 958\,\text{MeV}$). We also discussed the anomaly in 1+1 dimensional QFT. In our coming work, we will study the anomalies with bound state contributions by the Fujikawa [6] method.

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